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ABOUT THE RILLY'S METHOD OF CONSTRUCTION

OF 8x8 BIMAGIC SQUARES

ABSTRACT

This paper is a continuation of my previous paper about the "Revisit of the method of construction of the first bimagic squares" (ref. [1]).

I revisit here what I call "the Rilly's method" for generating 8x8 bimagic squares. This method was described in 1901 by Achille Rilly who found 230 bimagic squares of this type.

I built an enumeration program which showed that in fact there are more squares than 230. I pointed out also some little errors of Rilly.

HISTORICAL DATA

Achille Rilly is a Frenchman who is chiefly known for the first enumeration of the 38,039 bimagic series of order 8, in 1906 at a time when the computer was unknown (ref. [2]). But he worked also on the knight's tour problem and on the bimagic squares and stars (cf bibliography on p. 186 of the book of Cazalas "Carrés magiques au degré n").

Rilly published in 1901 a method of construction of particular 8x8 bimagic squares in a booklet ref. [3] which is difficult to find because it is not recorded in the general catalogue of the French National Library "Bibliothèque Nationale de France". Philippe Demonsablon told me he found this booklet in the Public Library of Troyes (France). This book has been digitalized by this Public Library and it is now possible to download it.

This original booklet describes the Rilly's method of construction but contains also an interesting *history of the discovery of the first bimagic squares*, more complete than the two references I gave in the p. 1 of my previous paper ref. [1].

Coccoz published in 1902 and 1903 two official communications (ref. [4] and [5]) to the AFAS about "the Rilly's method". Rilly himself published in 1907 to the AFAS a complementary note (ref. [6]). These three documents are on line.

DESCRIPTION OF THE RILLY'S METHOD

This method of construction first searches a semi-bimagic (SM) square and after makes permutations of rows and columns – like the Coccoz's method - , but the features of the SM squares are different. We can then state that the Rilly's method and the Coccoz's method are totally different.

The Rilly's method can be summarized as follows.

Instead using all the 38,039 bimagic series, the SM sq. of Rilly were built using the 2,704 bimagic series with 4 even numbers of sum 132

> *(I remind that any 8x8 bimagic series is made of 4 even numbers and of 4 odd numbers, with the sum of the even numbers being one of the 36 values 60, 64, 68,…, 200. Cf ref. [4] p. 137 and ref. [6] p. 42-43).*

For building his "generators" made of 8 bimagic rows, Rilly used 2 "half generators":

- a "superior half generator" for the 4 first rows with 32 given numbers:

    16 even numbers, from 2 to 16 and from 50 to 64

    16 odd numbers, from 17 to 47

    and each superior row has 2 of the highest even numbers 50, 52, 54,…, 64

- an "inferior half generator" for the 4 last rows with the 32 numbers which remain:

    16 even numbers, from 18 to 48

    16 odd numbers, from 1 to 15 and from 49 to 63

    and each inferior row has 2 of the lowest odd numbers 1, 3, 5,…,15.

Example of generator (cf ref. [4] p. 138):

64	50	14	4	45	33	27	23
62	52	16	2	41	37	31	19
60	54	10	8	47	35	25	21
58	56	12	6	43	39	29	17
48	36	26	22	57	55	11	5
46	34	28	24	61	51	15	1
44	40	30	18	59	53	9	7
42	38	32	20	63	49	13	3

The complements to 65 of the 32 numbers of the superior half generator are situated in the inferior half generator, and vice versa.

> In fact, it is possible to show that the 8 rows are made with only 136 special bimagic series out of 2,704: 68 bimagic series for the superior half generator, and 68 other

bimagic series for the inferior half generator, but these series are not necessarily complementary.

Rilly searched after, by permutations of columns, to obtain columns made with the 2,704 bimagic series. He built then a SM sq. Here is the SM sq #1 p. 139 of ref. [4] for the above mentioned generator:

64	33	4	27	45	14	23	50
37	62	31	2	16	41	52	19
8	25	60	35	21	54	47	10
29	6	39	58	56	17	12	43
11	55	22	48	26	36	57	5
51	15	46	24	34	28	1	61
18	44	9	53	59	7	30	40
42	20	49	13	3	63	38	32

In this SM sq., you can see there are not complementary lines as for the Coccoz's SM sq.

By permutations of rows and columns, a bimagic sq. is at last obtained (if possible). In the above mentioned example, it is the bimagic sq. #1 of Rilly (cf ref. [3] p. 133):

45	64	4	33	23	14	50	27
29	11	22	55	57	36	5	48
21	8	60	25	47	54	10	35
56	29	39	6	12	17	43	58
3	42	49	20	38	63	32	13
34	51	46	15	1	28	61	24
16	37	31	62	52	41	19	2
59	18	9	44	30	7	40	53

Rilly built by hand 50 superior half generators and 50 inferior half generators ; it is then possible to produce 50*50=2,500 generators. On these 2,500 generators, only 80 give SM sq. And on these 80 SM sq., Rilly calculated that only 40 give bimagic sq. Finally, Rilly built 230 bimagic squares which are printed in his booklet ref. [3].

ENUMERATION PROGRAM

I checked the results of Rilly by an enumeration program.

First I enumerated the possible bimagic series with the above mentioned conditions of Rilly. I found 68 bimagic series for each half generator. <u>These series are the same if the 32 numbers are only given</u> (it is not compulsory to write down the conditions about 2 of the highest even numbers or 2 of the lowest odd numbers).

After that I enumerated the possible SM sq. and the bimagic sq. I found more solutions than Rilly:

	Rilly	My program
Nb of half generators (sup or inf)	50	50
Nb of generators	2,500	2,500
Nb of generators giving SM sq	80	80
Nb of generators giving bimagic sq	40	48
Nb of SM sq	2,824	2,920
Nb of SM giving bimagic sq	230	477
Nb of ess. diff. bimagic sq	405	2,543

(I calculated the number 405 from the data of the booklet ref. [3] of Rilly, p. 120 to 132)

The differences are analysed in annex. The results of Rilly are not wrong, there are only <u>supplementary solutions which were not seen by Rilly. However some conclusions or assumptions of Rilly are wrong (see annex).</u>

Caution! I consider that 2 bimagic sq. are different if they are "essentially different", i.e. if they give 2 different sq. in standard position. <u>My definition of "different sq." is not the same</u> as the one of Coccoz or Rilly. For these authors, 2 sq. having the same rows and the same columns in a different order are the same sq.: they can be "presented in different manners" by permutation of rows/columns. Finally, the important thing for Coccoz and Rilly is the list of the SM sq. able to generate bimagic sq.

I made a lot of verifications of my numbers. A second enumeration by a different programmer should naturally be necessary for considering my numbers as sure.

<u>There are common squares given by the Coccoz's method and by the Rilly's method.</u> Among the 2,543 Rilly's sq., 1,521 are common to the Coccoz's type and to the Rilly's type ; 1,022 only are not common (we can call them "pure Rilly's squares"). With the 10,317 Coccoz's sq. and with the 2,543 Rilly's sq., we generate 10,317+1,022=11,339 ess. diff. sq.

I put in attachment the main files of my programs. I have the identification of each sq. in function of the source generator (and I have also the identification of the 11,319 in function of each Coccoz's or Rilly's sq.).

CONCLUSION

The enumeration of the Rilly's squares with a computer shows there are supplementary squares which were not seen by Rilly. It shows also that some conclusions or assumptions of Rilly are wrong.

But the method of construction is perfectly valid and we can cheer this man who built 230 squares without any computer!

After revisiting the Coccoz's method and the Rilly's method, I will deal in a future note with derived methods.

ACKNOWLEDGEMENTS

Thank you to Philippe Demonsablon who told me he found the booklet of Rilly in the Public Library of Troyes (France), and who analysed the Rilly's method in his book ref [7] on p. 505-506. This book of more than 1000 pages is a true encyclopaedia (with many original documents and contributions), chiefly about all the methods of construction which are analysed.

REFERENCES

[1] Revisit of the method of construction of the first magic squares, Francis Gaspalou, October 14, 2013 (email of the same date)

[2] Liste des 38 039 suites bimagiques de 8, Achille Rilly 1906 (Médiathèque du Grand Troyes, 10000 Troyes, fonds ancien, cote rés. K. 298).

[3] Etude sur les triangles et les carrés magiques aux deux premiers degrés, Achille Rilly 1901 (Médiathèque du Grand Troyes, 10000 Troyes, fonds ancien, cote rés. K. 17).

[4] Quelques exemples de carrés de huit magiques aux deux premiers degrés dont les lignes et surtout les diagonales sont de composition qui, n'étant point connues, n'ont pas été mentionnées en 1892 et 1893 aux congrès de Pau et Besançon, par M. V. Coccoz, Compte-rendu de la 31ème session de l'AFAS, Congrès de Montauban 1902, séance du 9 août 1902. Compte-rendu, seconde partie, p.137-157

[5] Carrés magiques, par M. le Commandant Coccoz, Compte-rendu de la 32ème session de l'AFAS, Congrès d'Angers 1903, séance du 6 août 1903. Compte-rendu, seconde partie, p.142-157

[6] Transformations dont sont susceptibles certains carrés bimagiques, par M. Achille Rilly, Compte-rendu de la 36ème session de l'AFAS, Congrès de Reims 1907, séance du 3 août 1907. Compte-rendu, seconde partie, p.42-48

[7] Histoire et anthropologie des carrés magiques, par Philippe Demonsablon (document provisoire du 30 avril 2010)

IN ATTACHMENT: zip file *with*:

1 list of the 2,543 sq (raw file)

2 list of the 2,543 sq (ordered file)

3 list of the 1,022 sq (pure Rilly)

4 list of the 11,339 sq (Coccoz or Rilly)

5 list of the 2,920 SM

6 list of the 2,500 generators

ANNEX: ANALYSIS OF THE DIFFERENCES

AND OTHER REMARKS

1. DIFFERENCES BETWEEN THE BOOKLET AND MY PROGRAM

1.1 <u>Some bimagic squares are overlooked by Rilly.</u> Example my bimagic sq # 387 (out of 2,543 in my raw file ; cf attachment 1):

2	52	16	62	27	41	21	39
48	11	34	5	24	51	26	61
28	63	22	49	36	7	46	9
23	37	25	43	14	64	4	50
13	42	3	40	53	18	59	32
35	8	60	31	58	29	33	6
54	17	45	10	47	12	56	19
57	30	55	20	1	38	15	44

This sq. comes from my generator # 432 (out of 2,500 ; cf attach. 6):

1	15	20	30	38	44	55	57
3	13	18	32	40	42	53	59
5	11	24	26	34	48	51	61
7	9	22	28	36	46	49	63
2	16	21	27	39	41	52	62
4	14	23	25	37	43	50	64
6	8	29	31	33	35	58	60
10	12	17	19	45	47	54	56

> (Note that in my program, I reversed the notions of “superior half generator” and “inferior half generator”, in comparison with the notations of Rilly).

This generator is the generator # $12^{s}39^{i}$ of Rilly (cf booklet p. 108 and 115).

On page 80 of the booklet, it is written that this generator # 12^s39^i gives 0 magic sq: it is wrong!

This generator is not in the Rilly's list of the 40 generators giving bimagic solutions, but it is in my list of the 48 generators giving bimagic solutions.

Note: here is the above mentioned bimagic square in standard position of Rilly

14	4	64	50	23	25	37	43
24	26	51	61	48	34	11	5
36	46	7	9	28	22	63	49
27	21	41	39	2	16	52	62
1	15	38	44	57	55	30	20
58	33	29	6	35	60	8	31
47	56	12	19	54	45	17	10
53	59	18	32	13	3	42	40

This square is not in the list of the 230.

1.2 More generally, most of <u>the numbers of solutions (SM and magic) coming from the generators in the table of Rilly p. 79-82 have to be revised.</u>

Ex. I: Rilly found 10 possible values for the number of SM: 1,4,7,12,25,27,29,35,43,303

I found 12 values: 1,4,7,12,25,29,34,43,44,304,310,342

Ex. II: it is written p. 80 that the generator # 12^s1^i gives 27 SM and 1 magic. I found it is in fact 29 SM and 6 magic (SM giving bimagic solutions, according to my vocabulary). The generator # 12^s1^i is my generator # 1582 (out of 2500) –cf attach. 6 -. The 6 SM giving bimagic solutions are my # 1508,1510,1520,1522,1527,1529 (out of 2920) – cf attach. 5 -.

Besides, I think that the Rilly's notion of "<u>corresponding generator</u>" (cf p. 77-78, remark II) is not very useful because 2 corresponding generators give surely the same number of SM solutions, but <u>not necessarily the same number of magic (as Rilly assumes it)</u>. Ex for the generators # 12^s1^i and 17^s2^i (which are my generators #1582 and #2092): they give each 29 SM but 6 magic for the one and 11 magic for the other. The table p. 79-82 has then to be totally changed!

> In two "corresponding generators", the groups of 4 even numbers and the groups of 4 odd numbers of each half generator are the same but in a different presentation.

On the contrary, the notion of "complementary generator" (cf p. 78-79, remark III) is perfectly valid and very interesting.

> In two "complementary generators", the complement to 65 of one superior half generator is the inferior half generator of the other, and vice versa.

Idem for the notion of "autocomplementary generator" which is defined p. 78: this notion is interesting. A little remark however: I think that Rilly doesn't make clearly the difference between the autocomplementary generators and the other ones. There are 2 kinds of generators: the autocomplementary generators and the other ones which work by complementary pairs. There are 50 autocomplementary generators among the 2,500, but only 10 among the 48 generators giving bimagic solutions. And there are (48-10)/2=19 pairs of complementary generators giving bimagic solutions.

1.3 All the long developments p. 83-99 about the search of diagonals become now useless with the systematic search by the computer.

2. OTHER REMARKS

Some other assumptions of Rilly are wrong.

2.1 Rilly assumes p. 72 that in one direction (row for example), there are only 2 kinds of groupments for the superior even numbers 50 to 64 in all the 8x8 bimagic squares:

- 2 even numbers by rows in 4 rows

- or 1 in each of the 8 rows.

It was true for the bimagic squares which were known at that time, but it is possible to find 8x8 bimagic squares where it is wrong. Ex. the bimagic sq. A1+H1=65 found by Walter Trump in July 2012:

64	17	15	62	24	21	22	35
49	37	39	3	12	58	46	16
41	26	44	7	29	50	4	59
34	20	13	38	27	8	57	63
32	61	40	42	5	11	51	18
14	60	53	30	52	19	9	23
25	6	2	47	56	45	43	36
1	33	54	31	55	48	28	10

In his booklet, Rilly deals with only the first category of squares and he claims p. 103 the credit for showing a general method for enumerating all the 8x8 bimagic squares. In fact there are truly other categories of 8x8 bimagic squares than those defined by Rilly. And the enumeration of all the 8x8 bimagic squares is still today an open question.

2.2 In his communication ref. [6] to the AFAS, Rilly has indicated 1+6 transformations working on all his 230 squares. These transformations - except the first one (transformation “complement to 65”) which is well known - don’t work on the new set of Rilly’s squares as defined now, i.e. they don’t work on all the 2,543 ess. diff. Rilly’s squares. For example, the 6 transformations don’t work on the square # 387 which is above mentioned at the § 1.1:

```
 2 52 16 62 27 41 21 39
48 11 34  5 24 51 26 61
28 63 22 49 36  7 46  9
23 37 25 43 14 64  4 50
13 42  3 40 53 18 59 32
35  8 60 31 58 29 33  6
54 17 45 10 47 12 56 19
57 30 55 20  1 38 15 44
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I remind the definition of the 1st of the 6 transformations: the even numbers are the same, and each odd number is replaced by its complement to 64.

Look at the first diagonal: the transformed diagonal by this transformation is not bimagic.

It is the same for the 5 other transformations.

The document ref. [6] is then obsolete.