Francis Gaspalou January 9, 2014

DERIVED METHODS FROM THOSE OF COCCOZ AND RILLY FOR GENERATING 8x8 BIMAGIC SQUARES

ABSTRACT

This paper is a continuation of my previous papers about the Coccoz's method (ref. [1]) and the Rilly's method (ref. [2]) for generating 8x8 bimagic squares.

I investigate here among the derived methods from those of Coccoz and Rilly. I had the idea of these methods after reading their publications.

I didn't drive complete enumerations of the sets of squares generated by these other methods – the task is too important - , I just explored the subject and showed examples. But with these derived methods, no doubt *it is possible to generate thousands of 8x8 bimagic squares!*

In annex, I revisit also the Coccoz's transformation which is a derived method found by Coccoz himself.

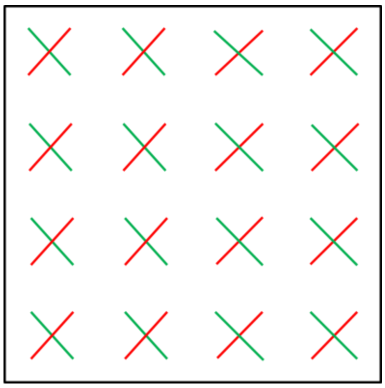
DERIVED METHODS FROM THE COCCOZ'S ONE

Pattern # 1 (Coccoz)

I remind an example of Coccoz's sq, with its source SM and with the pattern of this source SM:

2	23	48	57	35	13	54	28
21	4	59	46	56	26	33	15
52	37	30	11	17	63	8	42
39	50	9	32	6	44	19	61
16	25	34	55	45	3	60	22
62	43	20	5	31	49	10	40
27	14	53	36	58	24	47	1
41	64	7	18	12	38	29	51

1	58	36	27	53	14	24	47
15	56	46	21	59	4	26	33
22	45	55	16	34	25	3	60
28	35	57	2	48	23	13	54
40	31	5	62	20	43	49	10
42	17	11	52	30	37	63	8
51	12	18	41	7	64	38	29
61	6	32	39	9	50	44	19



It is the sq. # 1,054 of my list of 10,317 (cf ref. [1] attach. 1); the source SM with p=57 is on the p. 141 of the ref. [3].

By <u>pattern of a given bimagic square with “complementary lines”</u>, I mean the pattern of the source SM. It is a geometric drawing for the 16 subsquares with a total sum of 130 and with two numbers of sum p and two numbers of sum 130-p (p=33, 49, 57, 61, 63, 64 or 65). In two “complementary lines”, we need 4 couples of numbers of sum p and 4 other couples of numbers of sum 130-p.

In the definition of a pattern, permutations of rows or columns don’t matter: I consider it is the same pattern. Idem for the 8 symmetries of the square.

The notion of “pattern”- as above defined - is a generalization of the classical notion of “type”.

The Coccoz’s pattern has all the couples of sum p on parallels to one diagonal and all the couples of sum 130-p on parallels to the other diagonal. It is not the only one possible pattern. <u>We can imagine other patterns</u> and for each pattern, <u>an enumeration program is then possible</u> (with the 7 values for the parameter p), as for the Coccoz’s pattern. Here are some other patterns I found.

<u>For each example, I indicate the bimagic sq. in standard position, the source SM and the pattern of the source SM</u> (In fact, we can have sometimes several source SM with different p for a given bimagic sq., and here I indicate only one example of source SM).

I found these patterns by inspection of lists of bimagic sq.; I searched also “a priori” patterns.

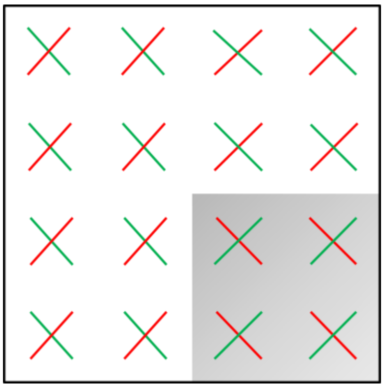
Note the limit of the method by inspection of lists of bimagic sq.: for most of the 8x8 bimagic sq., there are not “complementary lines” and it is then impossible to find a source SM pattern. Ex for the sq. # 166 of Rilly, ref. [6] p. 142 (which is also the sq. # 1 of my ordered list of 2,543 Rilly’s sq. - cf ref. [2] attach. 2 -).

Note also that for an “a priori” pattern, it is necessary to check that this pattern gives bimagic solutions. There are patterns which don’t give solutions. Ex: the axially symmetric pattern (cf ref. [7]).

<u>Pattern # 2</u>

5	57	48	22	36	26	55	11
43	12	58	39	17	56	6	29
50	23	27	4	14	45	33	64
61	1	24	46	28	34	15	51
10	47	35	60	54	21	25	8
40	30	53	9	7	59	44	18
32	38	13	49	63	3	20	42
19	52	2	31	41	16	62	37

1	28	15	46	51	24	61	34
38	63	20	49	42	13	32	3
57	36	55	22	11	48	5	26
30	7	44	9	18	53	40	59
23	14	33	4	64	27	50	45
52	41	62	31	37	2	19	16
47	54	25	60	8	35	10	21
12	17	6	39	29	58	43	56



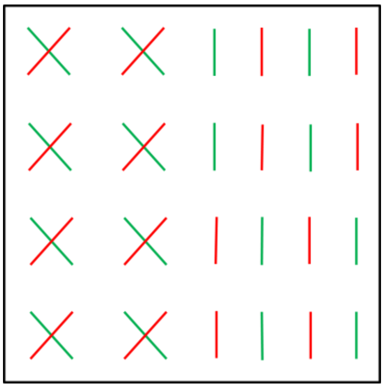
It is the sq. # 1 of Rilly in standard position (or my sq. # 2,164 of my ordered list of 2,543 Rilly's sq. ; cf ref. [6] p. 139 and ref. [2] attach. 2). The source SM is here with p=64.

The enumeration of the solutions with this pattern # 2 gives many bimagic squares like this one. For p=64 and with a trial program, I found more than 1,200 new sq., i.e. different from the 10,317 Coccoz's sq.

<u>Pattern # 3</u>

2	32	51	45	54	44	7	25
15	17	62	36	59	37	10	24
41	39	21	27	4	14	64	50
40	42	28	22	13	3	49	63
57	11	5	55	48	30	20	34
18	61	35	16	23	60	38	9
31	52	46	1	26	53	43	8
56	6	12	58	33	19	29	47

7	54	32	45	2	51	25	44
43	26	52	1	31	46	8	53
10	59	17	36	15	62	24	37
38	23	61	16	18	35	9	60
29	33	6	58	56	12	47	19
64	4	39	27	41	21	50	14
20	48	11	55	57	5	34	30
49	13	42	22	40	28	63	3



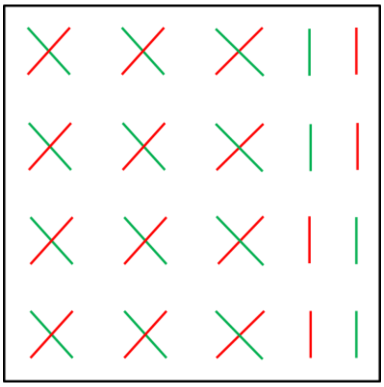
It is an "a priori" pattern. Here p=33.

The enumeration of the solutions with this pattern gives many bimagic sq.: in a trial limited program I found more than 700 bimagic sq. but all were already in the 10,317 Coccoz's list, except 4 new sq. like this one.

<u>Pattern # 4</u>

11	6	20	42	49	64	29	39
45	23	54	16	36	26	59	1
58	55	33	27	4	13	48	22
32	38	7	61	17	43	10	52
51	9	44	18	62	8	37	31
21	28	14	56	47	34	3	57
2	60	25	35	15	53	24	46
40	41	63	5	30	19	50	12

11	64	29	42	39	20	49	6
2	53	24	35	46	25	15	60
21	34	3	56	57	14	47	28
32	43	10	61	52	7	17	38
45	26	59	16	1	54	36	23
40	19	50	5	12	63	30	41
51	8	37	18	31	44	62	9
58	13	48	27	22	33	4	55



Here p=64.

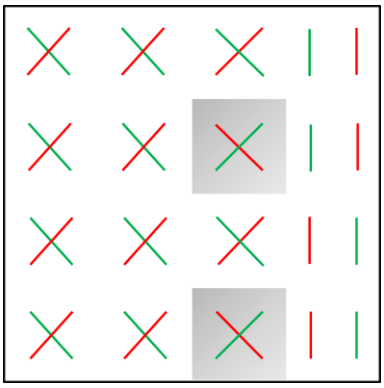
I found this sq. when enumerating the 362 bimagic sq. coming from a given generator on the field of the 2,704 bimagic series with 4 even numbers of sum 132.

It is not a Rilly's sq., but I found Rilly's sq. with this pattern. Ex the sq. # 1580 of my ordered list of 2,543 Rilly's sq. (cf ref. [2] attach. 2).

Pattern # 5

2	48	51	54	29	28	7	41
16	9	61	60	19	22	34	39
58	11	24	14	37	63	36	17
27	53	1	47	8	42	30	52
45	55	44	25	50	3	32	6
21	20	15	33	10	40	59	62
35	18	38	23	64	13	57	12
56	46	26	4	43	49	5	31

29	2	41	54	51	28	7	48
64	35	12	23	38	13	57	18
43	56	31	4	26	49	5	46
10	21	62	33	15	40	59	20
19	16	39	60	61	22	34	9
50	45	6	25	44	3	32	55
37	58	17	14	24	63	36	11
8	27	52	47	1	42	30	53

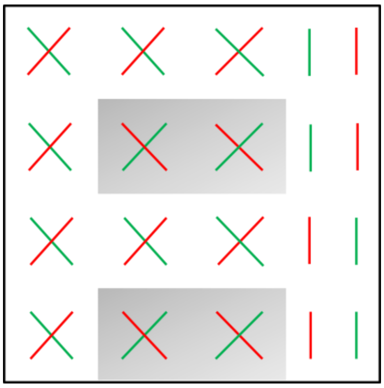


It is the sq. # 592 of my ordered list of 2,543 Rilly's sq. (cf ref. [2] attach. 2). Here p=64.

Pattern # 6

3	45	50	55	32	25	6	44
13	12	64	57	18	23	35	38
59	10	21	15	40	62	33	20
26	56	4	46	5	43	31	49
48	54	41	28	51	2	29	7
24	17	14	36	11	37	58	63
34	19	39	22	61	16	60	9
53	47	27	1	42	52	8	30

25	50	55	44	3	32	45	6
16	39	22	9	34	61	19	60
37	14	36	63	24	11	17	58
52	27	1	30	53	42	47	8
23	64	57	38	13	18	12	35
2	41	28	7	48	51	54	29
43	4	46	49	26	5	56	31
62	21	15	20	59	40	10	33

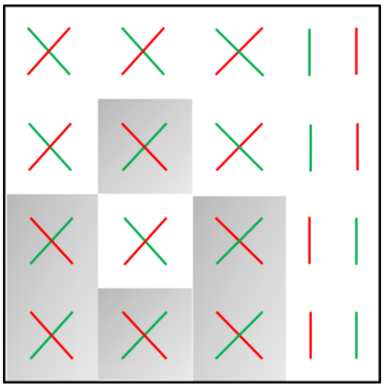


It is the sq. # 1338 of my ordered list of 2,543 Rilly's sq. (cf ref. [2] attach. 2). Here p=64 also.

Pattern # 7

3	36	15	48	61	30	49	18
44	14	20	54	39	1	31	57
45	11	21	51	34	8	26	64
6	37	59	28	9	42	56	23
55	24	10	41	60	27	5	38
32	58	40	2	19	53	43	13
25	63	33	7	22	52	46	12
50	17	62	29	16	47	4	35

25	44	55	6	45	32	3	50
22	39	60	9	34	19	61	16
33	20	10	59	21	40	15	62
46	31	5	56	26	43	49	4
52	1	27	42	8	53	30	47
63	14	24	37	11	58	36	17
12	57	38	23	64	13	18	35
7	54	41	28	51	2	48	29



For the SM, I made a rotation of 90° for having a pattern similar to the previous ones. Here p=64.

I found this sq. when enumerating the 361 bimagic sq. coming from an other given generator on the field of the 2,704 bimagic series with 4 even numbers of sum 132 (it is a different enumeration from the above mentioned one for the pattern # 4).

There are surely other possible patterns. For example, we can make the difference between the “basic patterns” (like the # 1, the # 3 or the # 4) and the “derived patterns” which are similar to these basic patterns, but the numbers inside some of the 16 subsquares are moved (these subsquares are shaded in the drawings). We remark that the basic patterns # 1 and # 4 generate derived patterns:

basic pattern # 1 → derived pattern # 2

basic pattern # 4 → derived patterns # 5, 6, 7

but not the basic pattern # 3 for the moment. We can then assume that the basic pattern # 3 generates also derived patterns to find.

Finally, it is surely possible to make with a computer a systematic search of all the 8x8 bimagic squares with “complementary lines” and with 16 subsquares as above defined.

DERIVED METHODS FROM THE RILLY’S ONE

I remind that the Rilly’s method works with 32 given numbers in one half generator:

16 odd numbers, from 1 to 15 and from 49 to 63

16 even numbers, from 18 to 48

(+ the 32 other remaining numbers - which are here their complement to 65 - in the other half generator).

The first set of 32 given numbers is not the only one possible.

For searching other sets of 32 numbers, I used the listing of the 841 ess. diff. associative 8x8 bimagic sq. (cf ref. [1] attach. 5) and I calculated the source generator for some associative bimagic sq. For example, the sq # 11 and its source generator (rows) are

2	7	46	43	52	53	32	25
11	14	39	34	57	64	21	20
60	61	24	17	10	15	38	35
49	56	29	28	3	6	47	42
23	18	59	62	37	36	9	16
30	27	50	55	48	41	4	5
45	44	1	8	31	26	51	54
40	33	12	13	22	19	58	63

1	8	26	31	44	45	51	54
2	7	25	32	43	46	52	53
3	6	28	29	42	47	49	56
4	5	27	30	41	48	50	55
9	16	18	23	36	37	59	62
10	15	17	24	35	38	60	61
11	14	20	21	34	39	57	64
12	13	19	22	33	40	58	63

They define a new set of 32 given numbers:

1 to 8, 25 to 32, 41 to 48, 49 to 56

With this new set of 32 numbers, I made an enumeration program, similar to the Rilly's program. I found a set of bimagic squares I called the Rilly* sq.:

	Rilly sq	Rilly*sq
Nb of bimagic series for rows	136	88
Nb of half generators (sup or inf)	50	28
Nb of generators	2,500	784
Nb of generators giving SM sq	80	52
Nb of SM sq	2,920	2,884
Nb of SM giving bimagic sq	477	450
Nb of ess. diff. bimagic sq	2,543	2,212

Cf the file in attach. 1 for the set of 2,212 Rilly* sq.

With the 11,339 Coccoz or Rilly sq, I found a total set of 11,916 sq. (cf attach. 2), i.e. there are only 577 sq. which are new (not Coccoz nor Rilly). I have the identification of each sq.

> Note 1: the set of 32 for the Rilly* sq. is coming from the generator of the sq # 11 (out of 841), but also from the generator of the sq # 213, or 400, or 401, or 402, etc. All these sq. are naturally associative sq inside the set of 2,212 Rilly* sq. (and they are also of Coccoz's type). With a filtering program, I found there are 106 associative ess. diff. such Rilly* sq among 2,212.

> Note 2: in the same way, the set of 32 for the Rilly sq can be considered as coming from the generators of the associative Rilly's sq. out of 841 (and these sq are also of Coccoz's type). I found there are 76 associative ess. diff. Rilly sq among 2,543. And there are 14 associative ess. diff. sq. which are both Rilly and Rilly* (Rilly for the rows and Rilly* for the columns or vice versa).

> Note 3: when looking at the source generator of the sq #11, we see the link with the "5 groupements" of Coccoz defined in the document ref. [3] p. 138-139. The set of 32 for the Rilly* sq is the same as the numbers of the 5^{th} "groupement" of Coccoz. The set of 32 for the Rilly sq is the same as the numbers of the 2^{nd} "groupement" of Coccoz. The "5 groupements" of Coccoz are not the only possible ones.

For searching other sets of 32 given numbers, we can try to use also the pandiagonal complete sq., the "5 groupements" of Coccoz, or even "a priori" sets. Numerous other sets of bimagic squares can then be surely computed with this method. But caution: most of these bimagic squares are maybe already known and then the number of truly new bimagic squares is not maybe very high.

Another idea of derived method is to search the solutions on the whole field of the 38,039 bimagic series and not only on the field of the 2,704 series with 4 even numbers of sum 132.

We can also put in the second half generator the numbers made with the complements to p or 130-p of the bimagic series situated in the first half generator (instead of the complements to 65). Ex with the rows #1,3,5,7 of the SM p=57 above mentioned on the p.1 on this note, we get the following set of 32 (made of 16 pairs of complements to 65):

> 1,3,5,7 ; 10,12,14,16,18,20,22,24 ; 25,27,29,31 and their complements to 65

> (we are sure that 4 bimagic series at least do exist with this set of 32).

CONCLUSION

One century ago, some Frenchmen found original methods for building 8x8 bimagic squares. With a computer I enumerated the exact number of squares generated by these methods. I investigated also among the derived methods and I showed that large supplementary sets of squares can be found. The subject is open and other discoveries are still possible.

But I think the total set of squares generated by these original and derived methods is only a very little part of the whole set of all the 8x8 bimagic squares.

REFERENCES

[1] Revisit of the method of construction of the first magic squares, Francis Gaspalou, October 14, 2013 (email of the same date)

[2] About the Rilly's method of construction of 8x8 bimagic squares, Francis Gaspalou, December 28, 2013 (email of the same date)

[3] Des carrés de 8 et de 9 magiques aux deux premiers degrés. Des carrés de mêmes bases en nombres triangulaires, par M. Coccoz, Compte-rendu de la 21ème session de l'AFAS, Congrès de Pau 1892, séance du 17 septembre 1892. Compte-rendu, seconde partie, p.136-148

[4] Des variations qu'on peut apporter aux carrés de huit magiques aux deux premiers degrés, par M. Coccoz, Compte-rendu de la 22ème session de l'AFAS, Congrès de Besançon 1893, séance du 4 août 1893. Compte-rendu, seconde partie, p.171-183

[5] Etude sur les triangles et les carrés magiques aux deux premiers degrés, Achille Rilly 1901 (Médiathèque du Grand Troyes, 10000 Troyes, fonds ancien, cote rés. K. 17).

[6] Quelques exemples de carrés de huit magiques aux deux premiers degrés dont les lignes et surtout les diagonales sont de composition qui, n'étant point connues, n'ont pas été mentionnées en 1892 et 1893 aux congrès de Pau et Besançon, par M. V. Coccoz, Compte-rendu de la 31ème session de l'AFAS, Congrès de Montauban 1902, séance du 9 août 1902. Compte-rendu, seconde partie, p.137-157

[7] The minimum order of an axially symmetric bimagic square is 12, Francis Gaspalou and Walter Trump, September 1, 2012

IN ATTACHMENT: zipped folder *with 2 files*:

1 list of the 2,212 Rilly* sq

2 list of the 11,916 Coccoz or Rilly or Rilly* sq

ANNEX: THE COCCOZ'S TRANSFORMATION

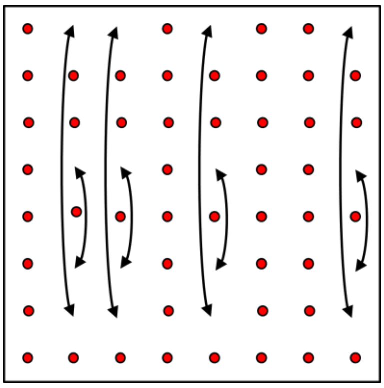
After publishing in 1892 his basic method of construction (ref. [3]), Coccoz described one year later (ref. [4]) what I call "the Coccoz's transformation". It is a transformation for <u>deriving another bimagic sq. from a given bimagic sq. by exchange of some cells in 2 or 4 parallel lines</u>.

For example, from the sq. Fig 1 in the paper ref. [4] p. 172:

38	47	3	10	29	24	60	49
32	21	57	52	39	46	2	11
7	14	34	43	64	53	25	20
61	56	28	17	6	15	35	42
51	58	22	31	12	1	45	40
9	4	48	37	50	59	23	30
18	27	55	62	41	36	16	5
44	33	13	8	19	26	54	63

we can exchange 16 cells in the 4 rows #1, 4, 6, 7 as follows:

38	27	55	10	41	24	60	5
32	21	57	52	39	46	2	11
7	14	34	43	64	53	25	20
61	4	48	17	50	15	35	30
51	58	22	31	12	1	45	40
9	56	28	37	6	59	23	42
18	47	3	62	29	36	16	49
44	33	13	8	19	26	54	63



and after make permutations of rows and columns for having magic diagonals:

38	27	5	60	10	55	41	24
32	21	11	2	52	57	39	46
44	33	63	54	8	13	19	26
18	47	49	16	62	3	29	36
61	4	30	35	17	48	50	15
7	14	20	25	43	34	64	53
51	58	40	45	31	22	12	1
9	56	42	23	37	28	6	59

It is the sq. Fig. 2 p. 175 of Coccoz (in fact, Coccoz gives only the sq. Fig. 1 and the sq. Fig. 2, he doesn't give the intermediate square ; the transformation is then hard to understand).

I remind that for Coccoz, two sq. having the same rows and the same columns in a different order are the same sq. For him, the intermediate sq and the final sq are the same sq.

In my language (cf my site), the Coccoz's transformation is equivalent to a geometric transformation between 8 couples of cells (in the initial sq. and in the intermediate SM sq.):

A2 and G2, A3 and G3, A5 and G5, A8 and G8

D2 and F2, D3 and F3, D5 and F5, D8 and F8

(4 transformations of order 2, twice)

For finding the lines where the cells can be exchanged, Coccoz searched the lines with 2 complementary pairs (in the transformed SM sq.) and used for that lists of bimagic series having this property (cf p. 175). All the 7 examples given by Coccoz in his paper ref. [4] have 4 rows with this feature (cf Fig. 2, 3, 4, 6, 7, 8, 9). I remind that this feature was useful at that time when the computer was unknown and was chiefly used for the search of the 2 diagonals.

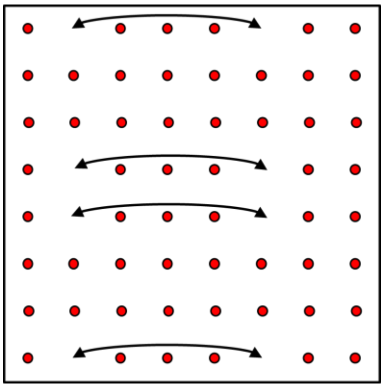
But it is not compulsory today to search only - as Coccoz did – the lines with 2 complementary pairs (in the transformed SM sq.): we can consider all the 38,039 bimagic series. It is possible also to find simpler transformations, with 8 cells exchanged only and without intermediate square. For example, I found the following transformation between 2 sq. from my list of 10,317 Coccoz sq. (cf ref. [1] attach. 1) in standard position

Sq. # 6,638

4	49	21	41	38	56	45	6
39	14	29	19	52	2	48	57
46	63	17	26	8	51	36	13
55	22	60	34	33	15	1	40
24	9	20	61	59	16	44	27
54	35	12	58	3	47	28	23
7	18	37	11	42	30	53	62
31	50	64	10	25	43	5	32

Sq. # 6,808

4	56	21	41	38	49	45	6
39	14	29	19	52	2	48	57
46	63	17	26	8	51	36	13
55	15	60	34	33	22	1	40
24	16	20	61	59	9	44	27
54	35	12	58	3	47	28	23
7	18	37	11	42	30	53	62
31	43	64	10	25	50	5	32



All the examples of sq. and of transformed sq. given by Coccoz in his paper are in my list of 10,317 sq.: there is not any new sq. However, I found transformed sq. (coming from my list of 10,317) which are new, i.e. out of this list.

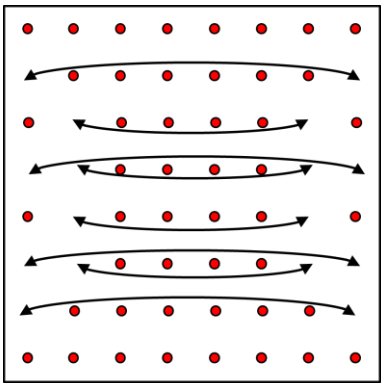
- Ex 1 where 16 cells have been exchanged:

Sq # 893

2	19	32	13	39	54	57	44
25	12	7	22	64	45	34	51
41	60	38	55	29	16	18	3
50	35	61	48	6	23	9	28
24	5	10	27	49	36	47	62
15	30	17	4	42	59	56	37
40	53	43	58	20	1	31	14
63	46	52	33	11	26	8	21

New sq

2	19	32	13	39	54	57	44
51	12	7	22	64	45	34	25
41	18	38	55	29	16	60	3
28	9	61	48	6	23	35	50
24	47	10	27	49	36	5	62
37	56	17	4	42	59	30	15
14	53	43	58	20	1	31	40
63	46	52	33	11	26	8	21



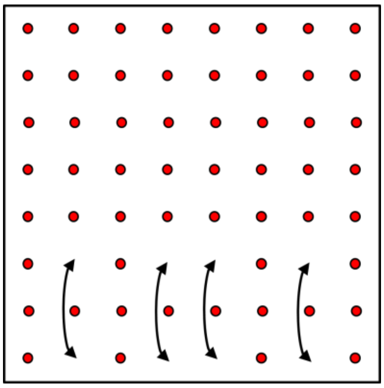
- Ex 2 where 8 cells have been exchanged.

Sq # 8,081

5	48	27	50	61	24	35	10
46	7	52	25	22	63	12	53
4	41	30	55	60	17	38	15
43	2	53	32	19	58	13	40
26	51	8	45	34	11	64	21
49	28	47	6	9	36	23	62
31	54	1	44	39	14	57	20
56	29	42	3	16	37	18	59

New sq

5	48	27	50	61	24	35	10
46	7	52	25	22	63	12	53
4	41	30	55	60	17	38	15
43	2	53	32	19	58	13	40
26	51	8	45	34	11	64	21
49	29	47	3	16	36	18	62
31	54	1	44	39	14	57	20
56	28	42	6	9	37	23	59



It should be surely possible to search more exhaustively these couples of solutions on the set of 10,317.

I think that the Coccoz's transformation uses the fact that there are many bimagic series (among the 38,039) which have special properties.

Example for the 4 transformations of order 2 (8 cells exchanged):

For the rows 6 and 8 of the last sq # 8,081, the conditions of this transformation are $F2+F4+F5+F7=H2+H4+H5+H7$ and $F2^2+F4^2+F5^2+F7^2=H2^2+H4^2+H5^2+H7^2$.

For the row 6, the bimagic series 6 9 23 28 36 47 49 62 and 3 16 18 29 36 47 49 62 have 4 common numbers.

For the row 8, idem for 3 16 18 29 37 42 56 59 and 6 9 23 28 37 42 56 59.

We have finally <u>4 bimagic series made with the same 4 quadruplets</u>

3 16 18 29, 6 9 23 28, 36 47 49 62, 37 42 56 59.

It is surely possible to enumerate all the groups of 4 bimagic series (among the 38,039) which have this property

In conclusion, the Coccoz's transformation seems to be only <u>the result of the inspection of lists of bimagic solutions:</u> inside these lists, we find several couples of solutions which are very close, i.e. which differ only from one another in some cells.