New Discoveries in the History of Magic Cubes

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Abstract

It is generally believed that the first perfect magic cube was presented by Frost in 1866. We show that the first such cube, of order 7, was actually already published in the year 1833 by Ferdinand Julius Brede alias de Fibre. We describe a possible construction method and give some information about the life of the discoverer. Additionally, we correct a misinterpretation of Andrew H. Frost's perfect magic cube of order 9 found in 1878.

Contents

1 Introduction

Magic cubes are 3-dimensional analogues of magic squares. They have already been discussed by Fermat, but the first examples of perfect magic cubes only date, as far as has been known until now, from the second half of the 18th century. For the history of magic cubes, see Boyer (2003) and Heinz (2004).

A *magic cube* of order $m > 1$ is an $m \times m \times m$-array of the numbers from 1 to m^3 such that the elements of each of the m^2 rows, m^2 columns, m^2 pillars, and four space diagonals all have the same magic sum, which is $S = \frac{1}{2}m \cdot (m^3 + 1)$. A magic cube is called *perfect* if the sum of m numbers along any straight line equals the magic sum. A property often considered in connection with magic cubes is *pandiagonality*, i. e. all broken diagonals also have the same magic sum.

Until now, it was believed that the first known perfect magic cube was found by Frost (1866). It had order 7. In this paper, we present an earlier such cube that was published in 1833 by a certain “de Fibre” alias F. J. Brede. In the following, we will first discuss Brede’s cube of order 7 and compare it to Frost’s. We will also briefly give some information on who Brede was. In the last part of the article, we will discuss the earliest perfect magic cubes of order 9. This will include a new interpretation of a cube that Frost presented in 1878.

2 de Fibre’s alias Brede’s perfect magic cube of order 7 from 1833

In the course of researches for potential additional material for his *Historical Dictionary of Mathematical Terms*, one of the authors was looking for occurrences of the German term *Zauberwürfel*, which is an alternative to the more common *magischer Würfel*. He found an obscure German magazine called *Iduna*, which was published in Hamburg and was directed at “young people of both sexes”, as its subtitle indicates. In this magazine (vol. 3, issues no. 4 and following, 1833), a certain “de Fibre” published, in weekly instalments, a perfect magic cube of order 7.

He also went on to publish a booklet on magic squares and magic cubes (Brede 1834). In the booklet, another perfect magic cube of order 7 is presented and it is discussed how such cubes can be constructed. The treatise has apparently never been mentioned in the mathematical literature; it is merely listed in a bibliography by Ahrens (1918, p. 389).

De Fibre’s cube from 1833 is shown in tables 1 to 4. The illustration shows the 7 plane sections through the cube perpendicular to the z-axis. The x-axis runs horizontally from left to right for each square and the y-axis runs vertically from top to bottom. The position of a cell of the cube is described by its coordinates (x, y, z) with $1 \leq x, y, z \leq 7$. $w(x, y, z)$ refers to the number in the cell (x, y, z).

Table 1: Layers 1 and 2

322	29	86	143	151	208	265
87	144	152	209	266	316	30
153	210	260	317	31	88	145
261	318	32	89	146	154	204
33	90	147	148	205	262	319
141	149	206	263	320	34	91
207	264	321	35	85	142	150

100	157	214	271	328	42	92
215	272	329	36	93	101	158
323	37	94	102	159	216	273
95	103	160	217	267	324	38
161	211	268	325	39	96	104
269	326	40	97	105	155	212
41	98	99	156	213	270	327

Table 2: Layers 3 and 4

277	334	48	56	106	163	220
49	50	107	164	221	278	335
108	165	222	279	336	43	51
223	280	330	44	52	109	166
331	45	53	110	167	224	274
54	111	168	218	275	332	46
162	219	276	333	47	55	112

62	119	169	226	283	340	5
170	227	284	341	6	63	113
285	342	7	57	114	171	228
1	58	115	172	229	286	343
116	173	230	287	337	2	59
231	281	338	3	60	117	174
339	4	61	118	175	225	282

Table 3: Layers 5 and 6

232	289	297	11	68	125	182
298	12	69	126	176	233	290
70	120	177	234	291	299	13
178	235	292	300	14	64	121
293	301	8	65	122	179	236
9	66	123	180	237	294	295
124	181	238	288	296	10	67

17	74	131	188	245	246	303
132	189	239	247	304	18	75
240	248	305	19	76	133	183
306	20	77	127	184	241	249
71	128	185	242	250	307	21
186	243	251	308	15	72	129
252	302	16	73	130	187	244

Table 4: Layer 7

194	202	259	309	23	80	137
253	310	24	81	138	195	203
25	82	139	196	197	254	311
140	190	198	255	312	26	83
199	256	313	27	84	134	191
314	28	78	135	192	200	257
79	136	193	201	258	315	22

2.1 Properties of the cube

The $7 \cdot 7 \cdot 7 = 343$ cells of the cube contain all integers from 1 to 343. The middle number 172 is in the centre of the cube.

In each orthogonal section, the sum of the 7 numbers in each row and each column is 1204 ($21 \cdot 14 = 294$ sums in total). In each orthogonal section, the sum of the 7 numbers in each of the two diagonals is 1204 ($21 \cdot 2 = 42$ sums in total).

The 4 space diagonals (often called *triagonals*) run through the centre and two corners of the cube. The sum of the numbers in the space diagonals is also 1204.

These properties represent the minimum requirement for a perfect cube. However, the degree of perfection of de Fibre's cube is higher.

De Fibre's cube is symmetrical (associated), i. e., the sum of the numbers from two cells whose connecting line runs through the centre of the cube and is bisected by the centre is 344.

De Fibre's cube is also pandiagonal, i. e., in each orthogonal section, the 7 numbers of each broken diagonal add up to 1204.

An even higher degree of perfection would be achieved if the broken triagonals were also magic. There are indeed cubes of order 7 that have this property, but they are not perfect, because some diagonals are not magic.

2.2 Construction principle of the cube

De Fibre constructed his cube using the superposition method. This method was found for magic squares as early as the 14th century by the Indian Narayana and later described in detail by Leonhard Euler. Whether de Fibre knew about it or re-invented it is not known (Brede 1834, p. 3 himself indicated that he did not actually read Euler; he only seems to have known that Euler dealt with magic squares). In any case, de Fibre transferred this method to three dimensions. He composed the cube from three components. Each component cube has the same dimensions as the cube he was looking for and contains the 7 characters a, b, c, d, e, f and g, each of which appears 49 times. All characters from a to g occur in each row, column, column and diagonal (including all broken diagonals). Ideally, this would also apply to the broken space diagonals. However, the cube has 8 corners and only 7 characters are available, so two corners contain the same character. The line connecting these two corners therefore cannot contain all 7 characters. This is the reason why in de Fibre's cube not all of the broken space diagonals are magic.

The construction principle is easier to recognize if each number is reduced by 1 and represented with the digits from 0 to 6 in the base-7 positional notation.

Table 5: Layer z=1 (left: C; middle: B; right: A)

6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3

3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0

6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3
0	1	2	3	4	5	6
3	4	5	6	0	1	2

Table 6: Layer z=2

2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6

0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4

1	2	3	4	5	6	0
4	5	6	0	1	2	3
0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4

Table 7: Layer z=3

5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2

4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1

3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3
0	1	2	3	4	5	6

Table 8: Layer z=4

1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5

1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5

5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3
0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1

Table 9: Layer z=5

4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1

5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2

0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3

Table 10: Layer z=6

0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4

2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6

2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3
0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5

Table 11: Layer z=7

3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0

6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3

4	5	6	0	1	2	3
0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0

2.3 A formula for de Fibre's cube

In the above illustration, the digits in the 3 positions are each shown in a separate cube. Instead of de Fibre's letters, we here use the digits from 0 to 6. The set of numbers $\{0, 1, 2, ...6\}$ forms the residue field modulo 7. In the following, all calculations are to be carried out modulo 7. For example, $6 + 4 = 3$ (7 must be subtracted or added until a number from 0 to 6 is found).

For component A, let $a(x, y, z)$ be the digit in cell (x, y, z). Similarly, we define $b(x, y, z)$ and $c(x, y, z)$. Each of the components A, B and C must be centrally symmetrical and therefore contain the number 3 in the centre: $a(4, 4, 4) = b(4, 4, 4) = c(4, 4, 4) = 3$.

The three components of de Fibre's cube have the following property, as do all other historical perfect $7 \times 7 \times 7$ cubes. The difference between the values of two neighbouring cells with a common side have the same value modulo 7 for a certain component A, B or C and a certain direction x, y, z. In each component, the value of a cell increases by 1 in the x-direction. This is also true if you consider $(1, y, z)$ to be the neighbouring cell of $(7, y, z)$. Note that $6 + 1 = 0 \bmod 7$ applies.

We can describe the increases in the x-direction by dx_A, dx_B and dx_C, all of which have the value 1.

In the y-direction we find: $dy_A = 3$, $dy_B = 2$, $dy_C = 2$. In the z-direction, the cells with the same x- and y-values in the individual squares must be traversed from top to bottom: $dz_A = 2$, $dz_B = 4$, $dz_C = 3$.

For all diagonals to be magic, the following must apply to each cube component:

$$dx + dy \neq 0 \bmod 7, dx - dy \neq 0 \bmod 7$$
$$dx + dz \neq 0 \bmod 7, dx - dz \neq 0 \bmod 7$$
$$dy + dz \neq 0 \bmod 7, dy - dz \neq 0 \bmod 7$$

The following matrix can therefore be used to describe the cube:

$$M = \begin{pmatrix} dx_A & dy_A & dz_A \\ dx_B & dy_B & dz_B \\ dx_C & dy_C & dz_C \end{pmatrix}$$

In addition, a translation vector t is required to ensure that the number 3 is in the centre. t can be calculated from M. The cube is therefore uniquely described by M.

We set $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = M \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + t \bmod 7$.

t is equal to the zero vector and can therefore be omitted if the sum of the three numbers in each row of the matrix is equivalent to 6 modulo 7. Collison's cube has this property (see section 2.4).

The formula for de Fibre's cube is $w(x, y, z) = 49c + 7b + a + 1$ with

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \bmod 7 \text{ and } 0 \leq a, b, c \leq 6.$$

2.4 Further pandiagonal associated magic $7 \times 7 \times 7$-cubes

de Fibre (1834): $\begin{pmatrix} 2 & 4 & 1 \\ 1 & 5 & 4 \\ 1 & 5 & 3 \end{pmatrix}$ with $t = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$

Frost (1866): $\begin{pmatrix} 1 & 5 & 3 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix}$ with $t = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

Langman (1962): $\begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 4 \\ 2 & 1 & 3 \end{pmatrix}$ with $t = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$

Benson & Jacoby (1981) $\begin{pmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 6 & 3 & 2 \end{pmatrix}$ with $t = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$

Collison (1990): $\begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ with $t = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Campbell (2008): $\begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{pmatrix}$ with $t = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$

Breedijk (2013 ?): $\begin{pmatrix} 1 & 4 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 5 \end{pmatrix}$ with $t = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

The equations that Collison presents for his cube are probably the most compact description of a perfect magic cube:

$$\begin{aligned} U &= 3x + y + 2z \bmod 7 \\ T &= 2x + 3y + z \bmod 7 \\ H &= x + 2y + 3z \bmod 7 \\ \text{where} \quad N &= 49H + 7T + U + 1. \end{aligned}$$

2.5 Differences between the cubes of Frost (1866) and de Fibre (1833)

For a simpler geometric description, the coordinate axes shall run through the centres of the cubes in the following. A comparison of the cubes yields:

- Component A of Frost = de Fibre's component B with 180° rotation around the x-axis

- Component B of Frost = de Fibre's component A with reflection on the xy-plane
- Component C of Frost = de Fibre's component C

The cubes are very similar, but it is unlikely that Frost copied from de Fibre. The construction method seems too obvious.

For comparison: Langman's cube is obtained from de Fibre's cube by simply swapping the x and y axes.

3 Ferdinand Julius Brede

"de Fibre" is a pseudonym made up of "F. J. Brede" (with $I = J$), the full name being Ferdinand Julius Brede. The little that is known about Julius Brede was summarized by Schmidt (1851, p. 1010), who remarks that Brede was reluctant to give information about his life. According to Schmidt, Brede was born in 1800 in Stettin (today Szczecin, Poland) and died on December 15, 1849, in Altona, leaving behind his wife Laura (née Masdorff) and three children; a fourth child was born after his death. Schmidt also remarks that Brede was an accountant and lists some of his publications. Overall, Schmidt gives the fullest account of Brede, but supplementary information can be found in church books and other sources.

According to the church books of Stettin (St. Jakobi), Ferdinand Julius Brede was born on May 18, 1800, as the eighth of nine children of Johann Christian Brede (1755-1834) and his wife Johanne Charlotte Julie Rauch (1765–1854). Julius' father came from Prenzlau, Brandenburg, and became a citizen of Stettin in 1781; he was a wine merchant and freemason and later served as Danish consul.

In 1824, Julius Brede is listed for the first time in the address book of Altona, now part of Hamburg. He worked there as an accountant for the rest of his life. On November 18, 1838, he married, also in Altona, the 19-year old "Laura Masdorff", who was actually baptized as Eleonore Catharina Margaretha Beutel. She was born on July 26, 1819, in Ober-Schmitten, Nidda. Why she later assumed the name Masdorff is unclear. The couple had five children altogether, one of which died early on. The two oldest children, Alma and Albert, later emigrated to Chile, where descendents of Brede still live.

Brede also began to publish chess compositions, which later led to a book on the game (Brede [1844]). His compositions were printed in various European magazines. He also devised a stenographical system (Brede 1827). Additionally, he wrote a number of poems, which were published in various magazines.

Among his more bizarre publications is a treatise on the Earth (Brede 1837), in which he claimed, according to the astronomer Heinrich Christian Schumacher (Peters 1863, p. 74), that the Earth was hollow inside. In a letter to Gauß, Schumacher described Brede as an "eccentric genius ... who writes about everything without understanding anything." However, Brede certainly *did* understand how to construct magic cubes.

4 Frost's perfect magic cube of order 9 from 1878

The first known perfect magic cube of order 9 has long been attributed to Planck (1905). However, a new analysis of Frost (1878) shows that the first such cube was actually found by Frost.

The magic $9 \times 9 \times 9$-cube from Frost's article in *The Quarterly Journal of Pure and Applied Mathematics* (Frost 1878, pp. 93ff.) has been misinterpreted by experts. Heinz (2004, p. 116), Hendricks (1992, p. 404) and others write that Frost's $9 \times 9 \times 9$-cube does not consist of consecutive numbers and is therefore not perfect. Frost (1878, p. 110) writes in his article: "... the numbers of nine parallel sections are given at W, and in the Nonary scale as exhibiting the order of the three digits ..."

In the presentation in Table W, he uses the base-9 positional notation. Then, of course, it would be good to reduce all numbers by one and use the numbers from $(000)_9$ to $(888)_9$. To avoid this reduction, Frost takes an unusual measure. As is usual in base-9 notation, he uses the digits 0 to 8 for the highest position with the value $9^2 = 81$ and the middle position with the value 9. However, Frost increases the last digit by 1 and thus uses the digits from 1 to 9. In total, his number notations then run from "001" to "889" (without the notations "xx0" and "x9x"). However, this unconventional representation can easily be converted into the decimal system: $(abc)_{Frost} = 81a + 9b + c$.

If Frost's representation is converted into the decimal system, the cube contains all numbers from 1 to $9^3 = 729$, so it is normal. In addition, it is perfect in every respect: not only all rows, columns, pillars, diagonals and triagonals but also all broken diagonals and triagonals are magic, and the cube is also centrally symmetrical (associated). For the representations of Frost's cube, see Trump (2024).

It is not unusual to represent cubes of order 9 in the positional notation with base 9. Benson and Jacoby do the same with their perfect $9 \times 9 \times 9$-cube. A representation of the cube in the decimal system cannot be found in their book *Magic Cubes: New Recreations* from 1981. Their cube contains the consecutive numbers from 0 to $(888)_9$ because, unlike Frost, they do not increase the last digit by 1. Nobody would think of saying that Benson and Jacoby's cube is not normal.

Frost's three-dimensional cube model from the Whipple Museum in Cambridge does not show two different cubes at the front and back, as assumed. They are exactly the same cubes, just written in different ways. One cube is rotated by 180° around the z-axis, so the corresponding numbers are opposite each other on the same mark.

In sum, in his article from 1878 Andrew H. Frost published an associated pandiagonal and pantriagonal magic $9 \times 9 \times 9$-cube. Until now, it was thought that Planck (1905) was the first who found such a magic cube.

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