Non-existence of central-symmetric perfect magic 5x5x5-cubes

The proof was found by **H. B. Meyer**, Germany in November 2012.

I just show the proof here with a different notation.

Definitions:

1) A **perfect magic cube** is a cubical array of positive integers from 1 to N^3 such that every straight line of N cells adds up to a constant.

   [*According to Benson and Jacoby, "Magic Cubes – New Recreations", 1981*]

   Note that in a perfect magic cube of order N all NxN-subsquares are magic squares.

2) Two cells of a magic cube are called a **central-symmetric pair** if the straight line from one cell to the other crosses the center of the cube and the distance from both cells to the center is the same.

3) A pair of numbers of a magic cube is called **complementary** if the sum of the two numbers is equal to the sum of the lowest and the highest number of the magic cube.

4) A **central-symmetric magic cube** is a magic cube where the two numbers of each central-symmetric pair of cells are complementary.

   (These cubes also were called associated or associative.)

Symmetric set of numbers

If you add or subtract a certain positive or negative integer to each number of a magic cube or a perfect magic cube or a central-symmetric cube then the new cube is still magic or perfect or central-symmetric.

To make things easier we use a symmetric set of consecutive numbers from –62 to 62.
Then the magic sum is 0 and also the sum of each complementary pair is 0.
The complement of number k is –k.

Notation of the entries: ixyz with the coordinates x, y, z = 1, 2, 3, 4, 5

Theorem: Central-symmetric perfect 5x5x5-cubes do not exist.

Proof by reductio ad absurdum

Assume that ixyz is a central-symmetric perfect 5x5x5-cube.

The two diagonals of the plane y = 2 are magic:

i121 + i222 + i323 + i424 + i525 = 0

i125 + i224 + i323 + i422 + i521 = 0

Subtraction of these two equations:

(A) i121 + i222 + i424 + i525 – i125 – i224 – i422 – i521 = 0

The same procedure for the two diagonals of plane x = 2:

i211 + i222 + i233 + i244 + i255 = 0

i215 + i224 + i233 + i242 + i251 = 0

Subtraction of these two equations:

(B) i211 + i222 + i244 + i255 – i215 – i224 – i242 – i251 = 0

(C) = (A) – (B)

i121 + i222 + i424 + i525 – i125 – i224 – i422 – i521 –
– i211 – i222 – i244 – i255 + i215 + i224 + i242 + i251 = 0

⇒

(C) i121 + i424 + i525 + i215 + i242 + i251 – i125 – i422 – i521 – i211 – i244 – i255 = 0

Delete sums of symmetrical pairs (like i424 + i242 = 0) ⇒

(C) i121 + i525 + i215 + i251 – i125 – i521 – i211 – i255 = 0

Replace entries of plane z = 5 by their complements (like i525 = – i141) ⇒

(C) i121 – i141 – i451 + i251 + i541 – i521 – i211 + i411 = 0

Arrange the terms in another order ⇒

(C) i121 + i251 + i541 + i411 – i521 – i211 – i141 – i451 = 0

Visualization of the equation (C):

	–		+	
+				–
–				+
	+		–	

Lemma: The equation (C) is true for each magic 5x5-subsquare of a central-symmetric perfect magic 5x5x5-cube. It is also true for the 6 oblique squares.

It is true for z=2 because the properties of the cube are invariant with respect to a 2–1–3–5–4–permutation of the planes when the permutation is done in all three directions x, y and z.

The equation is also true for z=5 and z=4 due to symmetries.

All statements for the planes orthogonal to the z-axis can also be made for the planes orthogonal to the x- or y-axis.

For the planes z=3, x=3 and y=3 and the 6 oblique squares the equation is trivially true because these squares are central-symmetric (associative).
(*The oblique squares do not play a role in this proof.*)

As we consider squares in different planes now, we leave away the coordinate for the plane:

(C) **i12 + i25 + i54 + i41 – i52 – i21 – i14 – i45 = 0**

(*The equation is also true when numbers from 1 to 125 were used.*)

We derive equation (D) by addition of the equations for the rows 1, 2, 4, 5 and subtraction of the equations for the two diagonals and for the column 3.

+	+	+	+	+
+	+	+	+	+
+	+	+	+	+
+	+	+	+	+

+

–		–		–
	–	–	–	
		– 3		
	–	–	–	
–		–		–

=

	+		+	
+				+
		– 3		
+				+
	+		+	

(D) i21 + i41 + i12 + i52 + i14 + i54 + i25 + i45 – 3 · i33 = 0

	–		+	
+				–
–				+
	+		–	

+

	+		+	
+				+
		– 3		
+				+
	+		+	

=

			+2	
+2				
		– 3		
				+2
	+2			

C + D = **E**

(E) = (C) + (D): **2 · i12 + 2 · i25 + 2 · i54 + 2 · i41 – 3 · i33 = 0**

Equation (E) is only correct if the center number i33 is even.

Corollary

The integer in the center of each magic 5x5-subsquare of a central-symmetric perfect magic 5x5x5-cube is even, when the numbers range from – 62 to + 62.

(*When the numbers range from 1 to 125, then the number in the center is odd*.)

That means that the row, the column and the pillar that run through the center of the cube consist of even integers only.

The complementary number of an odd number is odd because the sum of each complementary pair is 0.

Consider 12 rows: 5 of plane z=1, 5 of plane z=2 and the first two rows of plane z=3. Exactly half of all odd integers are in these 12 rows.

Number of odd integers in the 12 rows: 62 / 2 = 31

Thus the sum of all integers in the 12 rows is odd.

Contradiction:

The sum of all integers in the 12 rows is even,

because the sum of all integers in each row is 0.

Walter Trump, 2012-12-30