Theory of Magic Series Walter Trump, (c) 2002-09-08

This theory enables an inductive counting of the numbers of magic series.

Definition 1 (Kraitchik, 1942)

A set of **n** distinct integers taken from the interval **[1, n²]** form a **magic series of order n** if their sum is the **n**th magic constant $M_n = \frac{1}{2} n(n^2 + 1)$.

(For example {2, 8, 9, 15} is a magic series of order 4 since 2 + 8 + 9 + 15 = 34 .)

Definition 2

Let k, u, s be positive integers.
Define **N(k, u, s)** as number of sets $\{i_1, i_2, i_3, \dots, i_k\}$ of **k** distinct integers, that fulfill both of the following two conditions:

(1) $0 < i_1 < i_2 < i_3 < \dots < i_k \le \mathbf{u}$

(2) $i_1 + i_2 + i_3 + \dots + i_k = \mathbf{s}$.

Proposition 1

The number of magic series of order n equals $N(n, n^2, M_n)$

Proof: Follows directly from the definitions.

Proposition 2 (not essential)

$N(k, u, s) > 0 \Leftrightarrow \frac{1}{2} k(k+1) \le s \le \frac{1}{2} k(2u-k+1)$

Proof: Since condition (2) of definition 2 only can be fulfilled if s is not smaller than the sum of the first k natural integers and not greater than the sum of the last k integers of the interval [1, u] . You can find at least one proper set of integers if s meets the conditions on the righthand side.

Proposition 3 (not essential)

$s - \frac{1}{2} k(k-1) \le u_1, u_2 \Rightarrow N(k, u_1, s) = N(k, u_2, s)$

Proof: If u is large enough then the last integer i_k is maximal if $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, … , $i_{k-1} = k-1$. In this case the value of i_k equals $s - \frac{1}{2} k(k-1)$. If u is not less than this value then the number of sets is independent of u.

Theorem 1

Calculation of **N(k, u, s)** with k = 1 :

$u < s \Leftrightarrow N(1, u, s) = 0$
$u \ge s \Leftrightarrow N(1, u, s) = 1$

Proof: For k=1 there only exists one possible set $\{i_1\} = \{s\}$ that meets condition (2). Condition (1) is not fulfilled if u is less than s.

Theorem 2

Calculation of **N(k, u, s)** with k > 2 using values of the type **N(k–1**, x, y) :

$$N(k, u, s) = \sum_{w=k}^{\min(u, s-1)} N(k-1, w-1, s-w)$$

Proof: Take away the largest integer $w = i_k$ of a considered set then the remaining k–1 integers are from the interval [1, w–1] and sum up to s–w. We just have to add the values N(k–1, w–1, s–w) for all possible largest numbers $w \le u$. The summation starts with $w = k$ since the largest value of a set cannot be smaller than k. The summation ends at $w = u$ or at $w = s - 1$ to avoid that $(s - w)$ becomes zero or negative. [The summation may even end at $w = s - ½ k(k-1)$ if this value is smaller than u (Proposition 2).]

Algorithm

For fixed k the values of N(k, u, s) may be stored in one array say No(u, s). Fill the array with the values for $k = 1$ according to Theorem 1. Calculate the values of N(2, u, s) according to Theorem 2 starting with the largest s and store the results for all values of u in the same array. To avoid disturbations it is neccessary to start with the largest s and decrement s successively. Propositions 2 and 3 may be applied to decrease the amount of calculations. At the end of this procedure No(u, s) will contain the values of N(2, u, s). Continue with next k until you reach the desired order n. After each step the number of magic series of order k may be saved (Proposition 1).
Note that the size of the array depends on the highest considered order n, therefore declare $No(1 \ldots n^2, 1 \ldots M_n)$. If you want to get exact results, you have to use integer variables with high accuracy, in the case of $n = 32$ about 192 Bit per integer and about 400 MB for the complete array.

Results

Two <u>independent</u> calculations were done using 192-bit-integer and 64-bit-floatingpoint variables.

Order	Exact number of magic series	Floatingpoint value
01	1	1.00000000000000 E+00
02	2	2.00000000000000 E+00
03	8	8.00000000000000 E+00
04	86	8.60000000000000 E+01
05	1394	1.39400000000000 E+03
06	32134	3.21340000000000 E+04
07	957332	9.57332000000000 E+05
08	35154340	3.51543400000000 E+07
09	1537408202	1.53740820200000 E+09
10	78132541528	7.81325415280000 E+10
11	4528684996756	4.52868499675600 E+12
12	295011186006282	2.95011186006282 E+14
13	21345627856836734	2.13456278568367 E+16
14	1698954263159544138	1.69895426315954 E+18
15	147553846727480002824	1.47553846727480 E+20
16	13888244935445960871352	1.38882449354460 E+22
17	1408407905312396429259944	1.40840790531240 E+24
18	153105374581396386625831530	1.53105374581396 E+26
19	17762616557326928950637660912	1.77626165573269 E+28
20	2190684864446863915195866500356	2.19068486444686 E+30
21	286221079001041327793634043938470	2.86221079001041 E+32
22	39493409270082248457567923104977298	3.94934092700822 E+34
23	5739019677324553608481368828138484550	5.73901967732455 E+36
24	876085202984795348523051418634128837562	8.76085202984795 E+38
25	140170526450793924490478768121814869629364	1.40170526450794 E+41
26	23456461153390020211328759135664689342531028	2.34564611533900 E+43
27	4097641100787806775815644958425464097739938654	4.09764110078781 E+45
28	745947846718066619823209422870621836022069177558	7.45947846718067 E+47
29	141280774936453250057100993123755087750662375504136	1.41280774936453 E+50
30	27797610141981037322555479186167243505129073097363174	2.77976101419810 E+52
31	5673858009208148397135070998960708533898456476297052346	5.67385800920815 E+54
32	1199872454897380013845796517790093662180055383301098878668	1.19987245489738 E+57