On the Decomposition of the Neighborhood of a Planar Region into Congruent Parts

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With 4 Figures.

(English translation by ChatGPT and Walter Trump, 2025-03-17)

In the Annual Report of the D. M. V. (Deutsche Mathematiker Vereinigung), Volume 44 (1934), p. 41, Mr. K. Reinhardt posed the following problem (Problem 170): In the plane, two bounded (non-convex) congruent regions (e.g., two simple polygons) adjoin each other externally, leaving a gap between them. It is to be proven that this gap cannot be decomposed into parts congruent to the given congruent regions. (In some cases, a few can be fitted inside.)

The conjecture contained within this problem is actually incorrect, as I will demonstrate with an example. The type of example I will present is, apart from one other, demonstrably the only one that meets the required conditions, provided only one region is considered for filling.

By "region," I mean a simply connected bounded area of the (Euclidean) plane whose boundary is a Jordan curve. Two such congruent regions B_1 and B_2 may adjoin externally in such a way that a gap remains between them, forming a region B_0, and the question arises whether B_0 can be congruent to B_1 (or B_2). On the boundary C_0 of B_0, there are two points P and Q, which are boundary points of the regions B_0, B_1, and B_2. They divide C_0 into two arcs, b_{01} and b_{02}, which, apart from P and Q, contain only boundary points of B_0 and B_1 or of B_0 and B_2, respectively.

The following auxiliary theorem holds: If two congruent regions B' and B'' adjoin along an arc PQ, then B' and B'' transform into each other either

a) by rotation around the midpoint M of segment PQ through an angle, where the arc PQ must be point-symmetric with M as the center, or

b) by reflection along the line PQ, where the arc PQ coincides with the segment PQ, or

c) by one of the following transformations: α) rotation, β) parallel translation, or γ) glide reflection, where the arc PQ on B' (B'') corresponds to a proper sub-arc of the complementary arc to PQ on B'' (B'), or

d) by a glide reflection, where the arc PQ on B' (B'') transforms into an arc R''S'' (R'S') on B'' (B') that contains exactly one of the two points P or Q in its interior.

This auxiliary theorem can be proven through elementary considerations by examining the four possibilities: the arc PQ on B' corresponds under congruence to (1) a proper or improper sub-arc of PQ on B'', (2) a proper or improper sub-arc of the complementary arc of PQ on B'', (3) an arc on B'' containing either P or Q in its interior, or (4) an arc on B'' containing both P and Q in its interior.

Applying this auxiliary theorem to the figure consisting of the three regions B_0, B_1, and B_2, we arrive at the result that between B_0 and B_1, case a) or d) is possible, and simultaneously between B_0 and B_2, case c) α) is possible. All other combinations are impossible. Here, b_{01} is a point-symmetric or glide-symmetric (*) curve segment, and b_{02} is congruent to a curve segment that is a proper sub-arc of b_{01}.

(*) Probably an error, because in case d) b_{01} is not transformed into itself.

A polygon with the smallest number of vertices that belongs to the first type is, for example, the nonagon shown in Fig. 1 ($\varepsilon = 15°$). In this case, the boundary segment $ABCDEF$ is point-symmetric with respect to the center M; the segment $AIHGF$ is derived from $BCDEF$ by rotation around F through an angle. Three regions (A_0 ... I_0), (A_1 ... I_1), (A_2 ... I_2) can be assembled as required (Fig. 2). Here, $A_0 = F_1 = B_2$, $B_0 = E_1$, $C_0 = D_1$, $D_0 = C_1$, $E_0 = B_1$, $F_0 = A_1 = F_2$, $G_0 = E_2$, $H_0 = D_2$, $I_0 = C_2$.

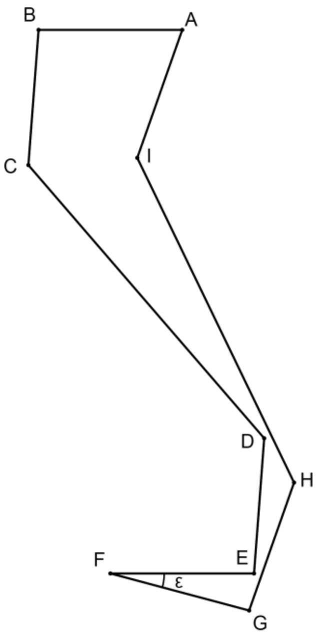

Fig. 1

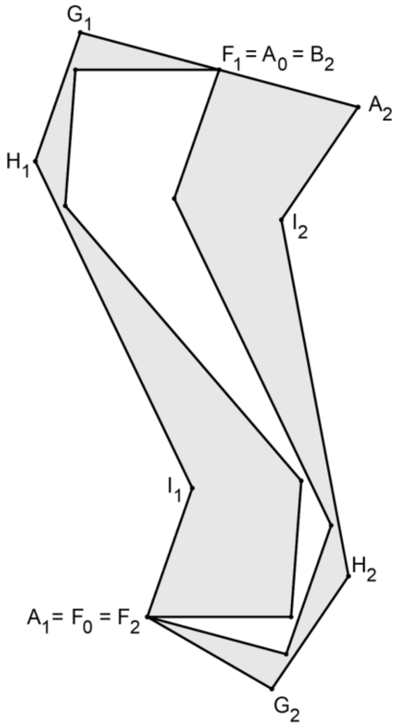

Fig. 2

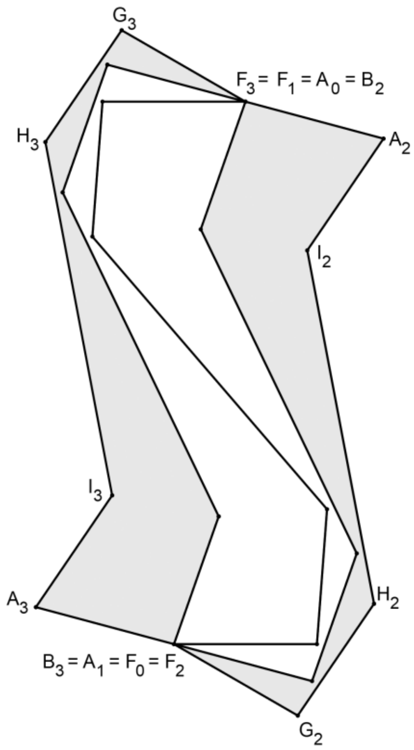

Fig. 3

This region possesses additional properties. It is possible to adjoin another nonagon (A_3 ... I_3) to (A_1 ... I_1) so that (A_3 ... I_3) and (A_2 ... I_2) form the gap, which is then filled by the two other nonagons (Fig. 3). Thus, such a gap is even decomposed into two given congruent regions.

Spiral Tiling of the plane

Furthermore, the nonagon serves as a decomposition region in Reinhardt's sense, meaning that it has the property that the entire plane can be tiled with congruent copies of it. Besides the regular tiling it induces, under suitable additional conditions, the entire plane can be tiled in a spiral form, as shown in Fig. 4, which is particularly remarkable.

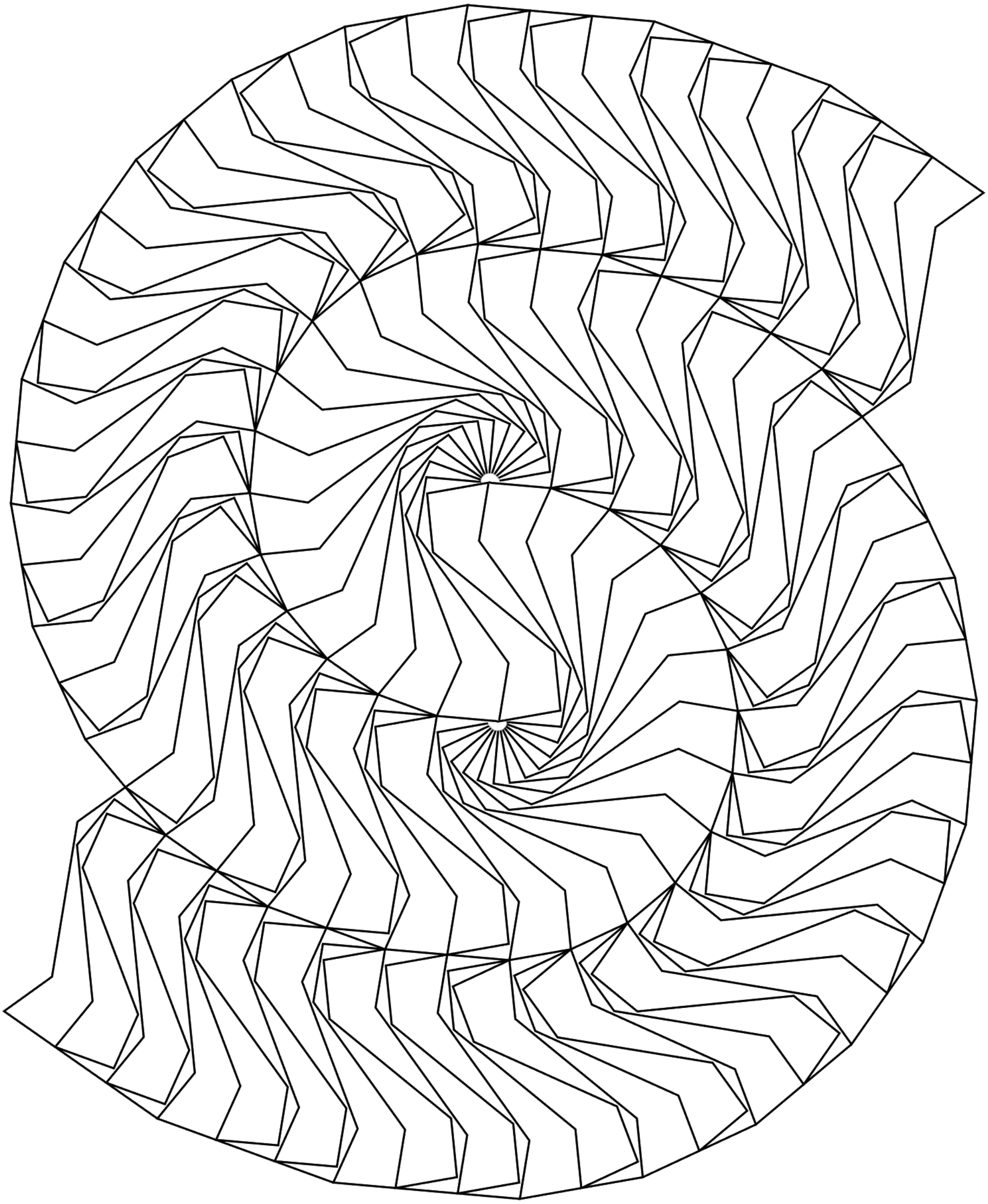

Fig. 4 (ε = 15°)

These investigations are of broader interest because they provide a small contribution to solving the general tiling problem – i.e., the problem of establishing all decomposition regions based solely on the concept of tiling. The complete solution to this problem is the subject of an ongoing study by Mr. Reinhardt, complemented by an additional investigation by the author, which will be published shortly.

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